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BOTH STATIC DEFLECTION AND VIBRATION MODE OF UNIFORM BEAM CAN SERVE AS A BUCKLING MODE OF A NON-UNIFORM COLUMN

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1. INTRODUCTION

There has been a long quest to find the closed-form solutions for vibration frequencies and the buckling loads of non-uniform structures. These have been pioneered 240 years ago by Euler [1] for the cross-section whose moment of inertia varies as

$$I(\xi) = I_0(a + b\xi)^m, \tag{1}$$

where I_0 is the moment of inertia at the origin of co-ordinates, $\xi = x/L$ is the non-dimensional axial co-ordinate, a and b are real numbers, m is the real number, chosen so that the moment of inertia is a positive quantity. Euler [1] studied two particular cases, namely $m = 2$ and 4, resulting in solutions in elementary functions, whereas Dinnik [2, 3] considered also the cases where $m \neq 2$ or 4, the solution being obtained in terms of Bessel functions. Dinnik [2, 3] reported some additional exact solutions, including exponentially varying moments of inertia.

In the discussion that ensued, Tuckerman [4] noted that Engesser [5] gave a way of obtaining an infinite number of closed-form solutions. Tuckerman [4] suggested to rewrite the buckling equation

$$EI(\xi)\,\mathrm{d}^2W/\mathrm{d}\xi^2 + PL^2W = 0 \tag{2}$$

as

$$I(\xi) = -PL^2W/EW'', \tag{3}$$

where prime denotes differentiation with respect to ξ. Substituting arbitrary function $W(\xi)$, that satisfies the boundary conditions, results in the desired variation of the moment of inertia $I(\xi)$. Engesser [5] discussed the case of parabolic deflection

$$W = cL(\xi - \xi^2) \tag{4}$$

which resulted in the moment of inertia

$$I(\xi) = 4I_0(1 - \xi^2), \tag{5}$$

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where I_0 is the moment of inertia in the middle cross-section, with the buckling load being $P_{cr} = 8EI_0/L^2$.

In this note, we construct an infinite set of closed-form solutions for two cases of buckling. The first is the Euler's case of buckling of simply supported columns whereas the second is the clamped-free column under its own weight. We pose a seemingly provocative question: *can a static deflection curve or a vibration mode of a beam serve as a buckling mode?* It is shown that the reply to this question is affirmative. The work on vibrating beams is underway and will be reported in the due course [6].

2. BASIC EQUATIONS

Consider the auxiliary problem of the simply supported uniform beam that is subjected to a distributed load of the intensity

$$p(\xi) = p_0 \xi^n. \tag{6}$$

The differential equation governing the static deflection $w(\xi)$ reads

$$EI\mathrm{d}^4 w/\mathrm{d}\xi^4 = L^4 p(\xi). \tag{7}$$

Integration and satisfaction of the boundary conditions results in the deflection

$$w(\xi) = \frac{\alpha L^{n+4}}{(n+1)(n+2)(n+3)(n+4)} \psi_1(\xi) \tag{8}$$

where

$$\psi_1(\xi) = \xi^{n+4} - \tfrac{1}{6}(n^2 + 7n + 12)\xi^3 + \tfrac{1}{6}(n^2 + 7n + 6)\xi \quad (n = 0, 1, 2, \ldots). \tag{9}$$

We have thus a countable infinity of functions that can be substituted into equation (3), to get an infinite sequence of distributions of elastic modulus, leading to the closed-form solutions.

3. BUCKLING OF NON-UNIFORM SIMPLY SUPPORTED COLUMNS

Demanding the buckling mode of the non-uniform column to coincide with the static deflection $\psi_1(\xi)$ of the uniform beam, we get from equation (3)

$$I(\xi) = -PL^2\psi/E\psi''. \tag{10}$$

We introduce a *parent moment of inertia* $I_p(\xi)$ as follows:

$$I_p(\xi) = -\psi_1/\psi''. \tag{11}$$

Then, equation (10) is rewritten as

$$I(\xi) = PL^2 I_p(\xi)/E. \tag{12}$$

For the function $I_p(\xi)$, we get the following expressions, listed here for the first 10 values of n:

$$n = 0: \quad I_p(\xi) = (1 + \xi - \xi^2)/12, \tag{13}$$

$$n = 1: \quad I_p(\xi) = 7/60 - \xi^2/20, \tag{14}$$

$$n = 2: \quad I_p(\xi) = \frac{1}{30}\left(\frac{4(1+\xi)}{(1+\xi+\xi^2} - \xi^2\right), \tag{15}$$

$$n = 3: \quad I_p(\xi) = \frac{1}{42}\left(\frac{6}{1+\xi^2} - \xi^2\right), \tag{16}$$

$$n = 4: \quad I_p(\xi) = \frac{1}{168}\left(\frac{25(1+\xi)}{1+\xi+\xi^2+\xi^3+\xi^4} - 3\xi^2\right), \tag{17}$$

$$n = 5: \quad I_p(\xi) = \frac{1}{72}\left(\frac{11}{1+\xi^2+\xi^4} - \xi^2\right), \tag{18}$$

$$n = 6: \quad I_p(\xi) = \left(14(1+\xi)\Big/\sum_{i=0}^{6}\xi^i - \xi^2\right)\Big/90, \tag{19}$$

$$n = 7: \quad I_p(\xi) = \left(52\Big/\sum_{i=0}^{3}\xi^{2i} - 3\xi^2\right)\Big/330, \tag{20}$$

$$n = 8: \quad I_p(\xi) = \left(21(1+\xi)\Big/\sum_{i=0}^{8}\xi^i - \xi^2\right)\Big/132, \tag{21}$$

$$n = 9: \quad I_p(\xi) = \left(25\Big/\sum_{i=0}^{4}\xi^{2i} - \xi^2\right)\Big/156, \tag{22}$$

$$n = 10: \quad I_p(\xi) = \left(88(1+\xi)\Big/\sum_{i=0}^{10}\xi^i - 3\xi^2\right)\Big/546. \tag{23}$$

Figure 1 represents the parent moments of inertia for values $n = 0$, 1, 2, 3 and 4, whereas Figure 2 portrays the functions for $n = 5$, 6, 7, 8, 9 and 10.

The immediate question arises: *what are the buckling loads?* To answer the question, we rewrite equation (12) as

$$P = EI(\xi)/L^2 I_p(\xi). \tag{24}$$

We substitute $I(\xi)$ by actual moment of inertia $I_a(\xi)$ to get

$$P = EI_a(\xi)/L^2 I_p(\xi). \tag{25}$$

Now, consider the actual moment of inertia $I_a(\xi)$ as being proportional to the parent moment of inertia $I_p(\xi)$ as

$$I_a(\xi) = \gamma I_p(\xi), \tag{26}$$

where γ is the coefficient of proportionality. Equation (24) becomes

$$P = \gamma E/L^2 \tag{27}$$

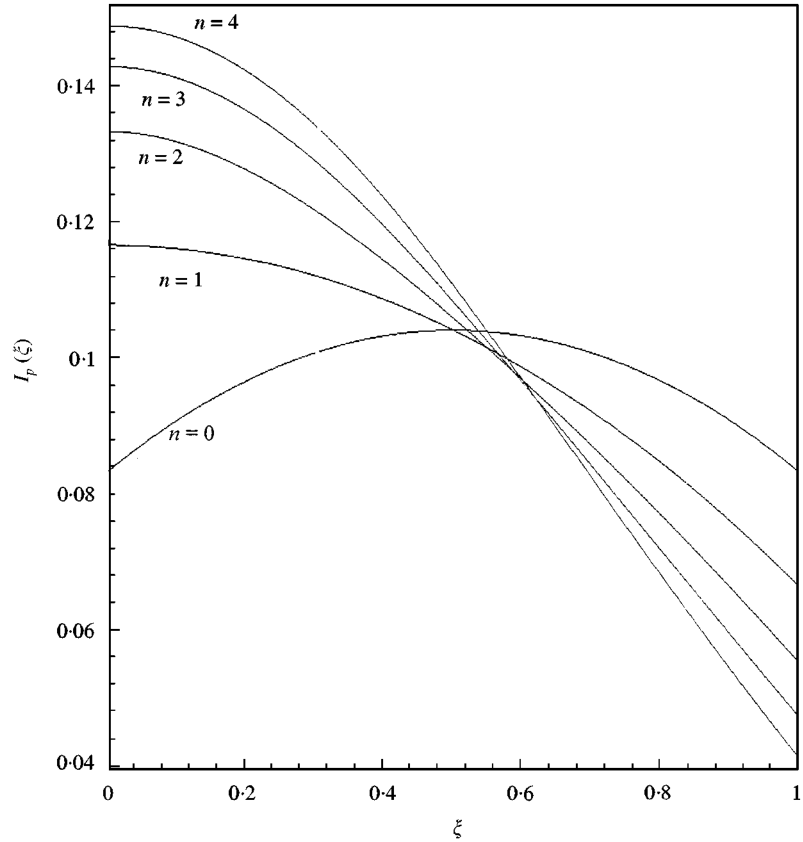

Figure 1. Parent stiffnesses for the simply supported column under compressive concentrated load, $n = 0, 1, 2, 3, 4$.

which is the expression for the buckling load. It depends on the arbitrary coefficient γ. Thus, through choosing γ at one's will one can achieve any pre-selected value of buckling load.

4. BUCKLING OF COLUMN UNDER ITS OWN WEIGHT

Consider now the buckling of the column under its own weight. The governing differential equation reads [7]

$$EI\mathrm{d}^2W/\mathrm{d}x^2 = \int_x^L q_0[W(u) - W(x)]\,\mathrm{d}u, \tag{28}$$

where q_0 is load intensity. Equation (28) can be rewritten as

$$IW'' = Q\left[\int_\xi^1 W(\gamma)\,\mathrm{d}\gamma - W(\xi)(1-\xi)\right], \tag{29}$$

where

$$Q = q_0L^3/E. \tag{30}$$

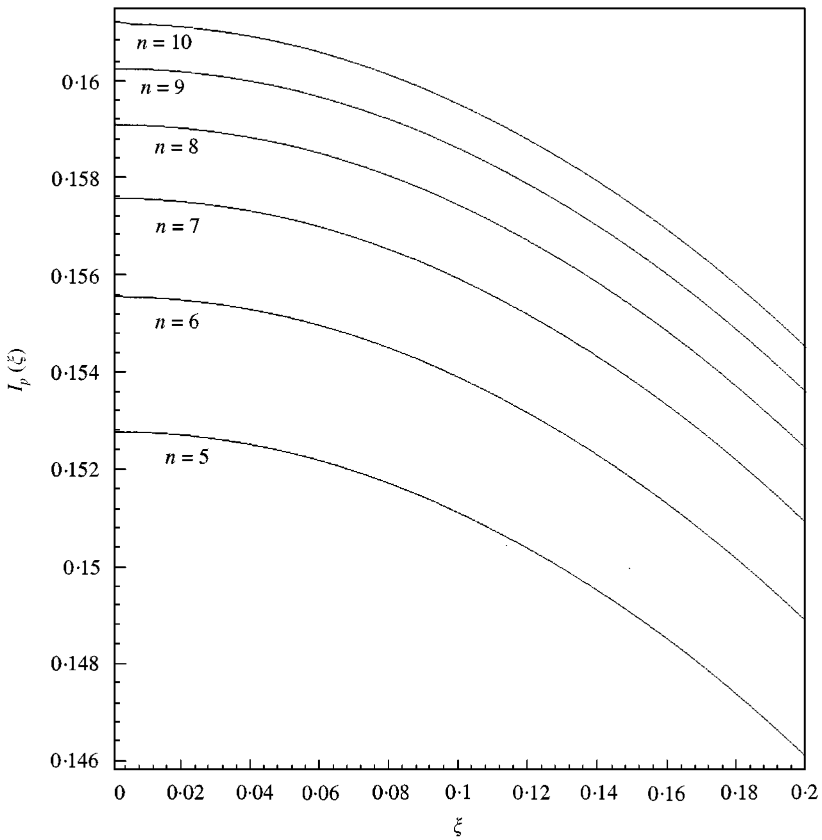

Figure 2. Parent stiffnesses for the simply supported column under compressive concentrated load, $n = 5, 6, 7, 8, 9, 10$.

We now resort to the auxiliary problem of the deflection of the uniform beam under the transverse load given in equation (6). For the clamped-free beam we get

$$w(\xi) = \frac{\alpha L^{n+4}}{(n+1)(n+2)(n+3)(n+4)}\,\psi_2(\xi), \tag{31}$$

where

$$\psi_1(\xi) = \xi^{n+4} - \tfrac{1}{6}(n+2)(n+3)(n+4)\xi^3 + \tfrac{1}{6}(n+1)(n+3)(n+4)\xi^2, \quad (n = 0,\ 1,\ 2, \ldots). \tag{32}$$

In new circumstances, we introduce the following parent moment of inertia:

$$I_p(\xi) = \left[\int_\xi^1 \psi(\gamma)\,\mathrm{d}\gamma - \psi(\xi)(1-\xi)\right]\Big/\psi''. \tag{33}$$

For various n, the parent moments of inertia read

$$n = 0: \quad I_p(\xi) = (3 + 6\xi - 6\xi^2 + 2\xi^3)/30, \tag{34}$$

$$n = 1: \quad I_p(\xi) = (26 + 52\xi - 42\xi^2 + 4\xi^3 + 5\xi^4)/120(2+\xi), \tag{35}$$

$$n=2: \quad I_p(\xi)=\frac{71+142\xi-102\xi^2+4\xi^3+5\xi^4+6\xi^5}{210(3+2\xi+\xi^2)}, \tag{36}$$

$$n=3: \quad I_p(\xi)=\frac{155+310\xi-207\xi^2+4\xi^3+5\xi^4+6\xi^5+7\xi^6}{336(4+3\xi+2\xi^2+\xi^3)}, \tag{37}$$

$$n=4: \quad I_p(\xi)=\frac{295(1+2\xi)-375\xi^2+4\xi^3+5\xi^4+6\xi^5+7\xi^6+8\xi^8}{504(5+4\xi+3\xi^2+2\xi^3+\xi^4)}, \tag{38}$$

$$n=5: \quad I_p(\xi)=\frac{511(1+12\xi)-627\xi^2+4\xi^3+5\xi^4+6\xi^5+7\xi^6+8\xi^8+9\xi^8}{720(6+5\xi+4\xi^2+3\xi^3+2\xi^4+\xi^5)}. \tag{39}$$

Figure 3 depicts the parent moment of inertia for values $n=0$, 1, 2, 3, 4 and 5, while Figure 4 gives a dependence of I_p as a function of ξ, for n taking values between 6 and 10.

We are again confronted with the question on the buckling load evaluation. In view of definition (30), and substituting $I(\xi)=I_a(\xi)$, equation (28) can be rewritten for the buckling intensity as

$$Q=q_0L^3/E=I_a(\xi)/I_p(\xi). \tag{40}$$

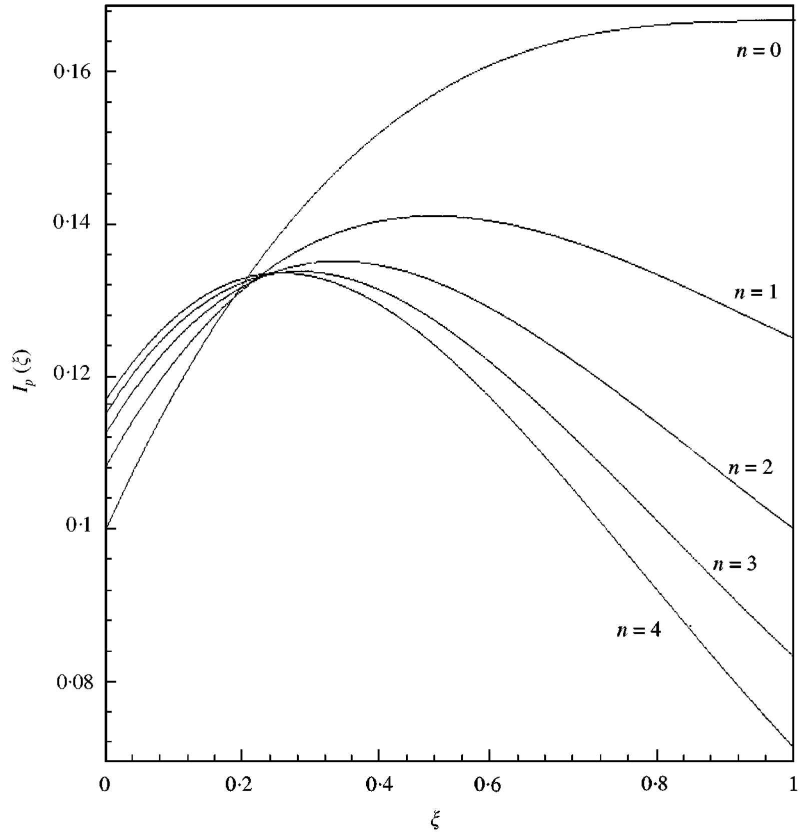

Figure 3. Parent stiffnesses for the clamped-free column under its own weight, $n=0$, 1, 2, 3, 4.

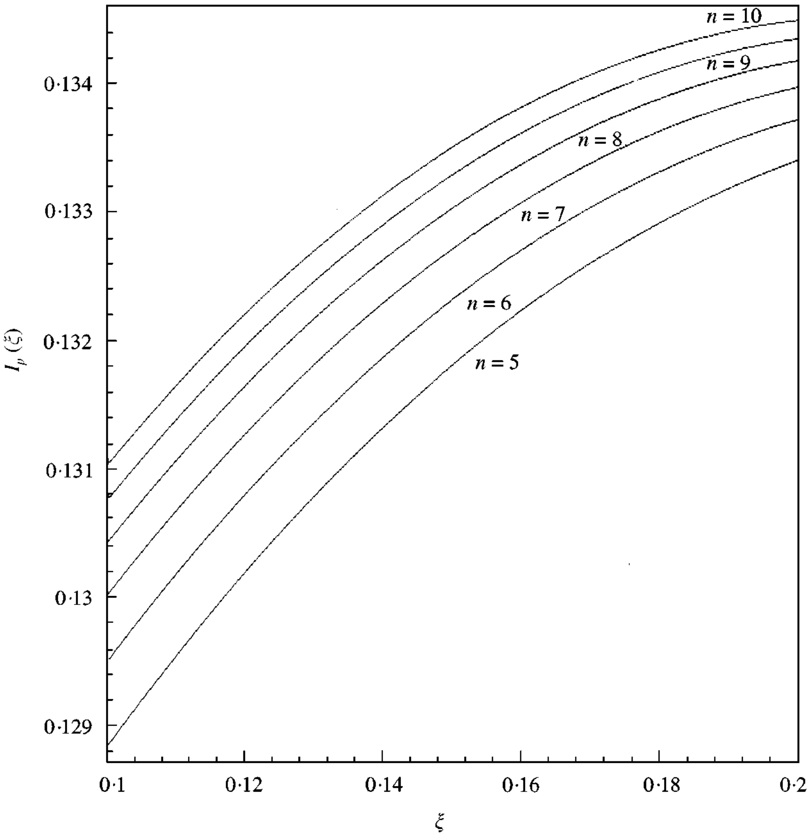

Figure 4. Parent stiffness of the clamped-free column under its own weight, $n = 5, 6, 7, 8, 9, 10$.

We choose the actual moment of inertia $I_a(\xi)$ to be proportional to $I_p(\xi)$,

$$I_a(\xi) = \delta I_p(\xi), \tag{41}$$

where δ is the coefficient of proportionality. Equation (40) takes the form

$$q_0 = \delta E/L^3. \tag{42}$$

Thus the buckling load can be made arbitrary by the proper choice of the parameter δ.

5. VIBRATION MODE OF AN UNIFORM BEAM AS A BUCKLING MODE OF A NON-UNIFORM COLUMN

Since the static deflections can serve as the buckling modes, it is natural to ask if vibration mode of an uniform beam can serve in such a capacity too. For the clamped-free uniform beam the fundamental vibration mode reads [8]:

$$\psi(\xi) = \sin\beta_1\xi - \sinh\beta_1\xi - r_1(\cos\beta_1\xi - \cosh\beta_1\xi), \tag{43}$$

where

$$r_1 = (\sin\beta_1 + \sinh\beta_1)/(\cos\beta_1 + \cosh\beta_1),$$

$$\beta_1 = 1{\cdot}8751040687119611664453082410782141625701117335311. \tag{44}$$

The reason for the extreme accuracy for β_1 will be explained later. The parameter β_1 satisfies the characteristic equation $1 + \cos(\beta_1)\cosh(\beta_1) = 0$. The calculation of the parent moment of inertia with this function, via equation (33) results in

$$I_p(\xi) = \frac{A}{\beta_1 M} - (1-\xi)\frac{B}{\beta_1^2 C}, \tag{45}$$

where

$$A = -\{[\cos(\beta_1)]^2 + 2\cos(\beta_1)\cosh(\beta_1) + [\cosh(\beta_1)]^2 + [\sin(\beta_1)]^2 - [\sinh(\beta_1)]^2$$
$$- 2\cos(\beta_1)\cosh(\xi\beta_1) - \cos(\xi\beta_1)\cosh(\beta_1) - \cosh(\beta_1)\cosh(\xi\beta_1)$$
$$- \sin(\xi\beta_1)\sinh(\beta_1) + \sinh(\beta_1)\sinh(\xi\beta_1)\}.$$

$$B = \sin(\xi\beta_1) - \sinh(\xi\beta_1) - \frac{[\sin(\beta_1)+\sinh(\beta_1)][\cos(\xi\beta_1)-\cosh(\xi\beta_1)]}{M}, \tag{46}$$

$$C = -\left[\sin(\xi\beta_1) + \sinh(\xi\beta_1) - \frac{[\sin(\beta_1)+\sinh(\beta_1)][\cos(\xi\beta_1)-\cosh(\xi\beta_1)]}{M}\right],$$

$$M = \cos(\beta_1) + \cosh(\beta_1).$$

Again, using equation (40).

$$Q = q_0 L^3/E = I_a(\xi)/I_p(\xi), \tag{47}$$

and choosing the actual moment of inertia

$$I_a(\xi) = \omega I_p(\xi), \tag{48}$$

we arrive at the buckling intensity

$$q = \omega E/L^3. \tag{49}$$

Due to arbitrariness of the positive parameter ω, the buckling parameter can be made as large as desired. Figure 5 depicts the parent moment of inertia: it should be borne in mind that for the accurate portrayal of it, there is a need of extreme accuracy for the parameter β_1, which is given in equation (44) with 50 significant digits. Otherwise, the parent inertial moment figure may present a seeming discontinuity in the vicinity of $\xi = 1$.

6. AXIALLY DISTRIBUTED NON-UNIFORM LOAD

Consider now the case studied by Dinnik [9] in terms of Bessel functions. The column is under axially distributed load proportional to ξ^t, where t is a positive integer. We utilize equation (28) with

$$q(x) = q_0(x/L)^t. \tag{50}$$

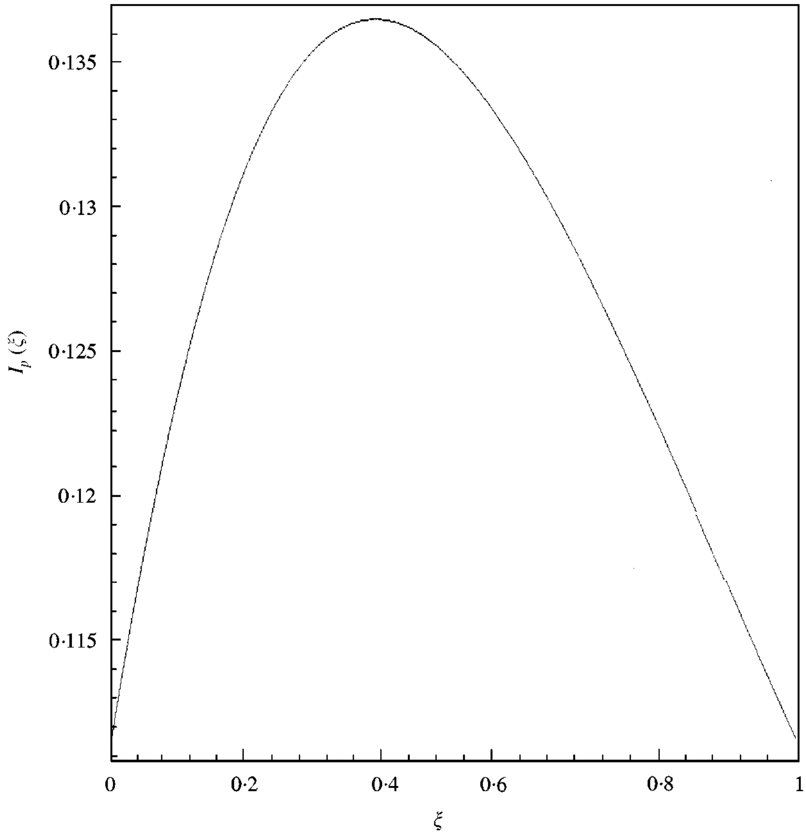

Figure 5. Parent stiffness of the vibrating non-homogeneous column with vibration mode of the uniform beam serving as a buckling mode of the non-uniform column.

Instead of equation (28) we have

$$\mathrm{I}\,\mathrm{d}^2w/\mathrm{d}x^2 = q_0 \int_x^L \left(\frac{u}{L}\right)^t [W(u) - W(x)]\,\mathrm{d}u$$
$$= q_0 \int_x^L \left(\frac{u}{L}\right)^t W(u)\,\mathrm{d}u - \frac{q_0 W(x)}{t+1}\left(L - \frac{x^{t+1}}{L^t}\right). \qquad (51)$$

With $u = \gamma L$ and $\xi = x/L$, equation (51) is rewritten as

$$I\,\mathrm{d}^2w/\mathrm{d}x^2 = Q\left[\int_\xi^1 \gamma^t W(\gamma)\,\mathrm{d}\gamma - \frac{W(\xi)}{t+1}(1 - \xi^{t+1})\right]. \qquad (52)$$

Introducing $W(\xi) = \psi(\xi)$ and defining the parent moment of inertia as

$$I_p(\xi) = \left[I_0(\xi) - \frac{\psi(\xi)}{t+1}(1 - \xi^{t+1})\right]\Big/\psi'', \qquad I_0(\xi) = \int_\xi^1 \gamma^t \psi(\gamma)\,\mathrm{d}\gamma, \qquad (53)$$

we get

$$I(\xi) = QI_p(\xi). \tag{54}$$

Choosing the actual moment of inertia $I(\xi) \equiv I_a(\xi)$ as being proportional to $I_p(\xi)$,

$$I_a(\xi) = \lambda I_p(\xi), \tag{55}$$

we obtain

$$q_0 = E\lambda/L^3. \tag{56}$$

For ψ we again use the vibration modes of the uniform clamped-free beams (32). The function $I_0(\xi)$ in equation (53) reads

$$I_0(\xi) = \int_\xi^1 \gamma^t \psi(\gamma)\,\mathrm{d}\gamma = \frac{R_0 + R_1\xi^{3+t} + R_2\xi^{4+t} + R_3\xi^{5+n+t}}{6(3+t)(4+t)(5+n+t)}, \tag{57}$$

where

$$R_0 = 432 + 822n + 495n^2 + 114n^3 + 9n^4 + 174t + 18t^2 + 2n^4t + 34n^3t + 2n^3t^2$$
$$+ 175n^2t + 15n^2t^2 + 317nt + 31nt^2,$$

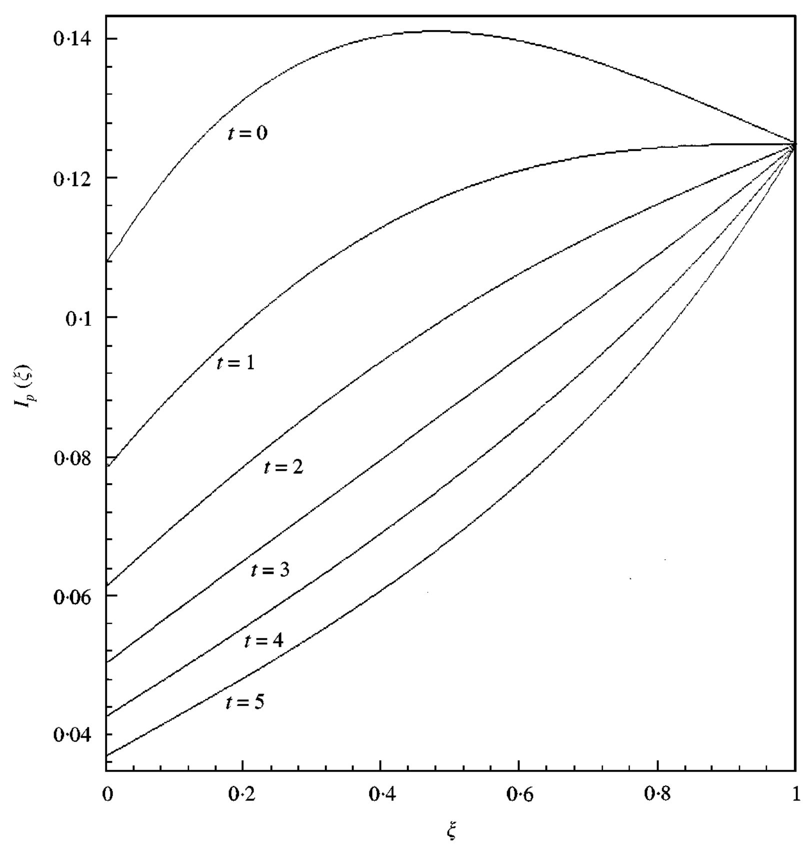

Figure 6. Parent stiffnesses for the clamped-free column under non-uniform axially distributed load, $t = 0, 1, 2, \ldots, 5$ and $n = 1$.

$$R_1 = -(720 + 1284n + 708n^2 + 156n^3 + 12n^4 + 324t + 36t^2 + 3n^4t + 51n^3t + 3n^3t^2 + 273n^2t + 24n^2t^2 + 549nt + 57nt^2), \quad (58)$$

$$R_2 = 360 + 462n + 213n^2 + 42n^3 + 3n^4 + 192t + 24t^2 + n^4t + n^4t + 17n^3t + n^3t^2 + 98n^2t + 9n^2t^2 + 232nt + 26nt^2,$$

$$R_3 = -(72 + 42t + 6t^2).$$

The analytical expressions obtained by the computerized symbolic algebraic code Maple when using as ψ the vibration mode of the uniform clamped-free beam are not reproduced here, due to their length. Some parent inertial moments are depicted in Figure 6 for various values of t ($t = 0, 1, 2, \ldots, 5$) and $n = 1$, while Figure 7 portrays $I_p(\xi)$ for $n = 2$. Figure 8 presented some parent inertial moment for various values of t ($t = 1, 2, \ldots, 5$) when we use for ψ the vibration modes of the uniform clamped-free beams (43).

7. CONCLUSION

We obtained several infinite series of closed-form solutions for the buckling loads of non-uniform columns. On the one hand, using infinite number of static deflections given in

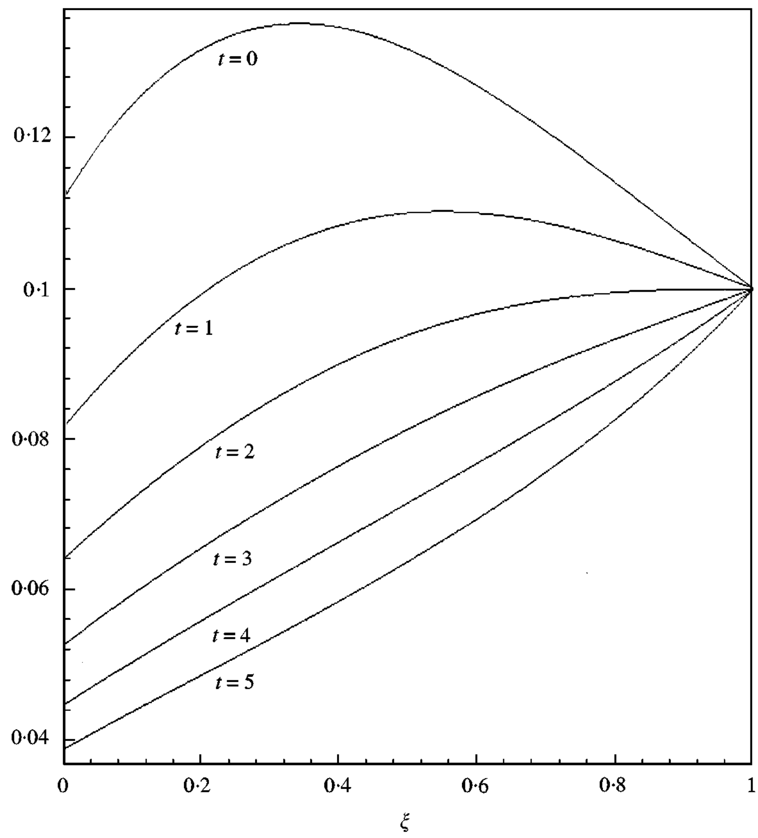

Figure 7. Parent stiffnesses for the clamped-free column under non-uniform axially distributed load, $t = 0, 1, 2, \ldots, 5$ and $n = 2$.

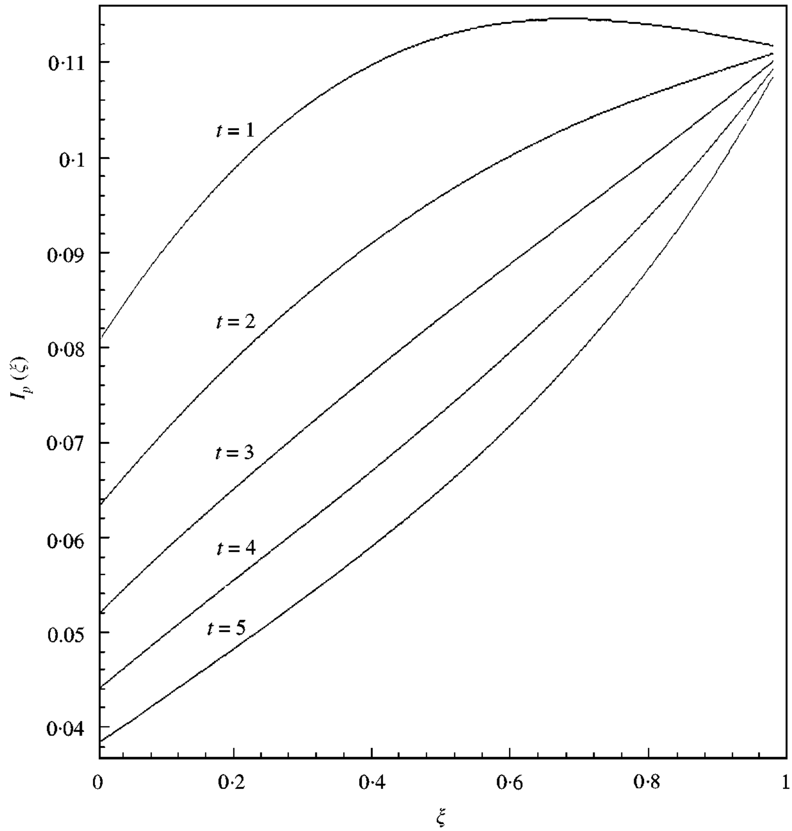

Figure 8. Parent stiffnesses for the vibrating non-homogeneous column with vibration mode of the uniform beam, $t = 1, 2, \ldots, 5$.

equation (8), we arrived at the infinite number of moments of inertia that lead to the postulated displacement, for the simply supported columns.

On the other hand, for the clamped-free columns, subjected to their own weight, an infinite number of closed-form solutions were obtained by using the postulated mode shapes in equation (32).

We also showed that the *vibration* mode of the uniform cantilever in equation (43) can serve as the exact *buckling* mode of the column under its own weight. For the column under axially distributed load proportional to axial co-ordinate in arbitrary positive power, we obtained, on the one hand, infinite number of closed-form solutions for any integer value of t on the other hand, an infinite number of solutions has been found for any n in equation (57) at any fixed value of t. A remarkable conclusion is that *all* solutions were obtained in exact closed-form manner. Moreover, the derived solutions appear to be attractive due to their *simplicity*. The present work provides numerous new exact closed-form solutions, in addition to those reported by Tuckerman [4], Duncan [10] and Elishakoff and Rollot [11].

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